

Fourteen new stationary points in the scalar potential of $SO(8)$ -gauged $\mathcal{N} = 8$, $D = 4$ supergravity

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Abstract

The list of six previously known nontrivial stationary points in the scalar potential of $\mathcal{N} = 8$, $D = 4$ supergravity with gauge group $SO(8)$ is extended by fourteen new entries, whose properties have been obtained numerically using the sensitivity backpropagation technique. Eight of the new solutions break the gauge group completely, while three have a residual symmetry of $U(1)$. Three further ones break the gauge group to $U(1) \times U(1)$. While the approximate numerical data are somewhat inconclusive, there is evidence that one of these may have a residual $\mathcal{N} = 1$ supersymmetry, hence correspond to a stable vacuum. It must be pointed out that this list of new solutions most likely is not exhaustive.

1 Introduction

The maximally supersymmetric field theory in four dimensions – $\mathcal{N} = 8$ supergravity [1] – that is obtainable by dimensional reduction of eleven-dimensional supergravity [2] is quite remarkable in many ways: The high degree of supersymmetry unifies the entire particle spectrum in a single supermultiplet, the model features an ‘unexpected’ hidden global exceptional $E_{7(7)}$ symmetry (see e.g. [3, 4]), and up to the possibility of a deformation that promotes the vector fields to nonabelian gauge bosons [5, 6, 7, 8], its structure seems to be uniquely determined. As the model with gauge group¹ $SO(8)$ arises from the compactification of $D = 11$ supergravity on a highly symmetric space – the seven-sphere S^7 – it occupies a prominent place in the family of supersymmetric field theory models, and as such it is not surprising to see that detailed information about its properties turned out to have a number of interesting applications [9, 10, 11, 12, 13]. Most of these are, of course, related to the AdS/CFT correspondence [14], as the vacua of $SO(8)$ -gauged $\mathcal{N} = 8$ supergravity correspond to stationary points in the scalar potential with negative (Planck-scale) cosmological constant, hence an Anti de-Sitter background geometry.

Considering the recent discovery that $\mathcal{N} = 8$ supergravity seems to be much more well-behaved at high loop orders than originally thought [15, 16], and has an especially simple S -matrix, it might even be justified to claim it to be “the simplest (interacting) quantum field theory” (implicitly: in four spacetime dimensions, and with a proper Yang-Mills kinetic term) [17].

¹This work uses the designations of nonabelian groups that are customary in supergravity throughout, even in cases where, strictly speaking, it would be more appropriate to denote them as a different quotient of the universal covering group

Given the prominence of four-dimensional $SO(8)$ -gauged $\mathcal{N} = 8$ supergravity, it is somewhat remarkable that little progress has been made on its vacuum structure during the last 25 years: While a detailed analysis by N. Warner in 1983 managed to produce an exhaustive list of all stationary points in the scalar potential with residual (unbroken) gauge symmetry of at least $SU(3)$ [18], and a further one that breaks $SO(8)$ to $SO(3) \times SO(3)$, [19], no new candidates for vacua of this model could be obtained despite repeated attempts (see e.g. [20, 21]). The underlying reason for this is easy to understand: The 70 scalars of the model can be thought of as parametrizing the coset space $E_{7(7)}/SU(8)$, so an analytic investigation of the full potential would seem to require an “Euler angle” type parametrization of that space. Even when taking the $SO(8)$ symmetry of the potential into account that reduces the number of degrees of freedom to $70 - 28 = 42$, the task of determining analytic expressions for such a parametrization of part of E_7 in terms of complex 56×56 matrices associated with its fundamental representation is easily shown to be well beyond technological reach: One would naively expect both the real and imaginary part of a typical matrix element to be a sum of contributions containing a sine or cosine factor in each coordinate, so a reasonable estimate for the number of individual trigonometric factors needed to represent each matrix element is $2 \cdot 42 \cdot 2^{42} \approx 4 \cdot 10^{14}$. This must be regarded as a very conservative lower estimate, considering that the occurrence of spinorial $SO(8)$ representations in the definition of E_7 means that there will be more options than just a $\sin(\alpha)$ or $\cos(\alpha)$ factor for each angle coordinate (the angle can occur with a number of different half-integral coefficients): Clearly, a full analytic treatment of the problem hence is technically unfeasible. What has been done so far to nevertheless extract some information about the symmetry breaking structure of this potential was to use a group-theoretic argument to reduce the analysis to specially chosen more tractable low-dimensional sub-manifolds of $E_7/SU(8)$ that are invariant under some subgroup of the gauge group $SO(8)$. If this subgroup G is chosen sufficiently large, or if a suitable embedding is chosen, the G -invariant submanifold \mathcal{M}_G may allow a manageable analytic parametrization.

While this strategy allowed the determination of a number of nontrivial solutions, and by far has not been exploited to its full theoretical potential yet, considering the large number of possible options for G such that \mathcal{M}_G is low-dimensional and easily parametrized, this approach seems to be somewhat unrewarding, as for many specific choices of residual symmetry, the potential by no means has to have associated stationary points. Still, it seems reasonable to expect the number of stationary points with almost completely – or even completely – broken gauge symmetry to be fairly large. For this reason, other, complementary, techniques must be considered to study the extremal structure of the scalar potentials of models of supergravity – this, as well as a number of others. As has been demonstrated in [22], the “reverse mode algorithmic differentiation” method, also known as “sensitivity backpropagation”, is a stunningly potent numerical technique that allows an extremely effective analysis of these problems: For the case involving the largest exceptional Lie group E_8 , namely $\mathcal{N} = 16$ ‘Chern-Simons’ supergravity in three dimensions with gauge group $SO(8) \times SO(8)$, dozens of new solutions could be determined with very little effort. It hence makes sense to also apply this technique to gauged $\mathcal{N} = 8$, $D = 4$ supergravity.

2 The scalars of $SO(8)$ -gauged $\mathcal{N} = 8$ SUGRA

The deformation of maximal supergravity in four dimensions that promotes the 28 vector fields to nonabelian gauge bosons of $SO(8)$ requires the introduction of a scalar potential at second order in the coupling constant g , in order to maintain supersymmetry. Due to the emergent global E_7 symmetry of maximal four-dimensional supergravity, this potential is most easily expressed as a particular sixth-order polynomial function of the entries of an E_7 ‘56-bein’ associated with the coset manifold of scalars $E_{7(7)}/SU(8)$. This function has a local maximum where all scalar fields are set to zero, which is associated with unbroken $SO(8)$ gauge symmetry and full $\mathcal{N} = 8$ supersymmetry in Anti-deSitter space with a negative cosmological constant of $V = -6g^2$. In addition to this, there are a number of further saddle points in the potential, some of which nevertheless are stable due to the strong gravitational back-reaction in such a spacetime with Planck-scale cosmological constant: as long as the energy gained by leaving a stationary point is over-compensated by the energy that has to be expended in the kinetic term for a localized variation, the solution is perturbatively stable. This stability criterion is encoded in the Breitenlohner-Freedman bound [23] on the lowest mass of the scalar excitations:

$$\frac{m_{\min}^2/g^2}{-V/g^2} \geq -\frac{9}{4} = -2.25. \quad (1)$$

When listing a number of stationary points with highly broken gauge symmetry, such as here, it makes sense to spell out the conventions on E_7 matrices in full detail and then refer to this common basis, rather than discussing each case in terms of the types of scalar fields that get excited and their interrelation. This is done in the rest of this section.

2.1 The $E_{7(+7)}$ Lie algebra

The Lie algebra of the group $E_{7(7)}$ allows a beautiful triality-symmetric construction based on the subalgebra $so(8)$: Just as $so(8)$ can be extended to $so(9)$ by adding eight extra generators that transform in the eight-dimensional vector representation $\mathbf{8}_v$, plus the obvious commutators for these new generators amongst one another, we can also perform a triality-symmetric extension of $so(8)$ to f_4 by simultaneously adding the three inequivalent real eight-dimensional vector (V), spinor (S), and co-spinor (C) representations, again with the obvious commutator relations, which here are e.g. of the form $[V, S] = C$ and involve the $so(8)$ Clifford algebra generators $\gamma_{\beta\gamma}^\alpha$. If one instead suitably extends $so(8)$ by the symmetric traceless matrices over the vectors, spinors, and co-spinors, $\mathbf{35}_v$, $\mathbf{35}_s$, $\mathbf{35}_c$, one obtains e_7 . Signs and factors of i can be chosen in such a way that the generators obtained from $\mathbf{28} + \mathbf{35}_v$ give rise to the Lie algebra of the maximal compact subgroup, $su(8)$, while $\mathbf{35}_s + \mathbf{35}_c$ give the 70 ‘boost’ generators of $E_{7(7)}$. In terms of the maximal compact $SU(8)$ subgroup of $E_{7(7)}$, the $\mathbf{35}_s$ and $\mathbf{35}_c$ correspond to the self-dual and anti-self-dual four-forms.

A similar triality-symmetric construction of $e_{8(8)}$ starts from the subalgebra $so(8) \times so(8)$ and extends this with three 64-dimensional representations that transform as $(\mathbf{8}_v, \mathbf{8}_v)$, $(\mathbf{8}_s, \mathbf{8}_s)$, and $(\mathbf{8}_c, \mathbf{8}_c)$. The maximal split form is obtained by choosing signs in such a way that $so(8) \times so(8) + (\mathbf{8}_v, \mathbf{8}_v)$ form the Lie algebra of the maximal compact subgroup, $so(16)$.

Representation	Indices	Range
$so(8)$ vector	a, b, \dots	$1 \dots 8$
$so(8)$ spinor	α, β, \dots	$1 \dots 8$
$so(8)$ co-spinor	$\dot{\alpha}, \dot{\beta}, \dots$	$1 \dots 8$
$so(8)$ $\mathbf{35}_s$	$(\alpha\beta), (\gamma\delta) \dots$	$1 \dots 35$
$so(8)$ $\mathbf{35}_c$	$(\dot{\alpha}\dot{\beta}), (\dot{\gamma}\dot{\delta}) \dots$	$1 \dots 35$
$su(8)$ vector	$i, j, \dots; I, J, \dots$	$1 \dots 8$
$su(8)$ adjoint	A, B, \dots	$1 \dots 63$
e_7 fundamental	$\mathcal{A}, \mathcal{B}, \dots$	$1 \dots 56$
e_7 adjoint	$\mathcal{A}, \mathcal{B}, \dots$	$1 \dots 133$
None (counting index)	$\mathbf{a}, \mathbf{b}, \dots$	Situation-dependent

Table 1: Index alphabets

It is useful to note that, when taking a diagonal $so(8)$ subalgebra of $so(8) \times so(8)$, these 64-dimensional representations each decompose as $\mathbf{28}_{\text{anti}} + (\mathbf{1} + \mathbf{35})_{\text{symm}}$, and the diagonal $so(8)$ together with the three $\mathbf{35}$ give $e_{7(7)}$, while the three singlets form the $sl(2)$ of the maximal subalgebra $e_{7(7)} \times sl(2)$ of $e_{8(8)}$.

From this discussion, we see that explicit generators of E_7 that are constructed along these lines are slightly more awkward to work with than the corresponding generators of E_8 , the reason being that the most obvious and convenient choices of a (rational) basis for the $\mathbf{35}_{v,s,c}$ representations are non-orthonormal. This also has direct implications for strategies to simplify the presentation of stationary points that have been found numerically.

In the following, we will use different alphabets to designate different irreducible representations. Note that for some non-fundamental representations, we use composite symbols which nevertheless are supposed to be read as single glyphs. The notation may at times seem unusually tedious, but was chosen with the objective in mind to simplify transliteration of formulas to computer code as much as possible – with the exception of *not* adopting the convention that index counting starts at zero, which, while highly desirable from a technical perspective, would be too much in disagreement with established practice in physics. The alphabets listed in table 1 will be employed. We also occasionally use primes to enlarge alphabets, hence both a and a' will be considered to be $so(8)$ vector indices. Furthermore, composite indices such as $[ijk]$ are used for lexicographically enumerated 3-forms.

The “typewriter indices” $\mathbf{a}, \mathbf{b}, \dots$ that occasionally show up serve a dual purpose: on the one hand, they allow us to take some short-cuts, such as when defining index range decompositions. In particular, they are employed e.g. to avoid the otherwise common ‘matrix block notation’ for decomposing the complex 56×56 matrices that represent $E_{7(7)}$ group elements, which may be dangerous here due to possible factor-2 ambiguities. Their second purpose is to keep expressions computer friendly and allow a more or less straightforward transliteration of these formulas to computer code.²

²If index counting uniformly starts at zero, rather than one, this makes conversion of tensor multi-indices to linear indices much more straightforward. Hence, it generally is advisable to adhere to this convention in computer calculations, and shift indices when producing L^AT_EX output.

Also, we will frequently make use of the ‘leicographical index splitting function’ $Z(\mathbf{n})$ that maps a number \mathbf{n} in the range $1 \dots 28$ to an index pair (i, j) , $1 \leq i < j \leq 8$ such that 1 gets mapped to (1, 2), 2 to (1, 3), 7 to (1, 8), 8 to (2, 3), etc.: $Z((i-1) \cdot 8 + j - i(i+1)/2) = (i, j)$.

The detailed construction of 133 complex 56×56 generator matrices that satisfy the e_7 commutator algebra is given in Appendix A.

2.2 The scalar potential

Parametrizing the coset manifold of scalars $\mathcal{H} = E_7/SU(8)$ by 70 boost generator coefficients $\phi_{\mathbf{n}}$, the potential $V(\phi_{\mathbf{n}})$ of $SO(8)$ -gauged $\mathcal{N} = 8, D = 4$ supergravity is a simple quadratic expression in the tensors A_1, A_2 , which are defined in terms of the so-called T -tensor that is (in $D = 4$) cubic in entries of the exponentiated generator $\exp(\sum_{\mathbf{n}} \phi_{\mathbf{n}} g^{(\mathbf{n})})$, specifically:

$$\begin{aligned}
V/g^2 &= -\frac{3}{4} A_1^{ij} (A_1^{ij})^* + \frac{1}{24} A_2^i{}_{jkl} (A_2^i{}_{jkl})^* \\
\text{with:} & \\
A_1^{ij} &= -\frac{4}{21} T_m{}^{ijm} \\
A_2^{\ell ij} &= -\frac{4}{3} T_{\ell}{}^{i'j'k'} \delta_{i'j'k'}^{ijk} \\
T_{\ell}{}^{kij} &= (u^{ij}{}_{IJ} + v^{ij}{}_{IJ}) (u_{\ell m}{}^{JK} u^{km}{}_{KI} - v_{\ell m}{}^{JK} v^{km}{}_{KI}) \\
\\
\mathcal{V}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{A}}} &= \exp(\sum_{\mathbf{n}} \phi_{\mathbf{n}} g^{(\mathbf{n})})^{\tilde{\mathcal{A}}}{}_{\tilde{\mathcal{B}}} \\
u_{ij}{}^{IJ} &= 2 \mathcal{V}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{A}}} \delta_{\tilde{\mathcal{A}}}^{\mathfrak{m}} \delta_{\tilde{\mathcal{B}}}^{\mathfrak{B}} \delta_{ij}^{\mathfrak{ab}} \delta_{cd}^{IJ} \\
&\text{for } \tilde{\mathcal{A}} \leq 28, \tilde{\mathcal{B}} \leq 28, (a, b) = Z(\mathfrak{m}), (c, d) = Z(\mathbf{n}) \\
u^{kl}{}_{KL} &= 2 \mathcal{V}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{A}}} \delta_{\tilde{\mathcal{A}}}^{\mathfrak{m}} \delta_{\tilde{\mathcal{B}}}^{\mathfrak{B}} \delta_{ab}^{\mathfrak{kl}} \delta_{KL}^{\mathfrak{cd}} \\
&\text{for } \tilde{\mathcal{A}} > 28, \tilde{\mathcal{B}} > 28, (a, b) = Z(\mathfrak{m} - 28), (c, d) = Z(\mathbf{n} - 28) \\
v_{ij}{}_{KL} &= 2 \mathcal{V}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{A}}} \delta_{\tilde{\mathcal{A}}}^{\mathfrak{m}} \delta_{\tilde{\mathcal{B}}}^{\mathfrak{B}} \delta_{ij}^{\mathfrak{ab}} \delta_{KL}^{\mathfrak{cd}} \\
&\text{for } \tilde{\mathcal{A}} \leq 28, \tilde{\mathcal{B}} > 28, (a, b) = Z(\mathfrak{m}), (c, d) = Z(\mathbf{n} - 28) \\
v^{kl}{}^{IJ} &= 2 \mathcal{V}_{\tilde{\mathcal{B}}}^{\tilde{\mathcal{A}}} \delta_{\tilde{\mathcal{A}}}^{\mathfrak{m}} \delta_{\tilde{\mathcal{B}}}^{\mathfrak{B}} \delta_{ab}^{\mathfrak{kl}} \delta_{cd}^{IJ} \\
&\text{for } \tilde{\mathcal{A}} > 28, \tilde{\mathcal{B}} \leq 28, (a, b) = Z(\mathfrak{m} - 28), (c, d) = Z(\mathbf{n})
\end{aligned} \tag{2}$$

In this work, all anti-symmetrizers are normalized to work as projectors, e.g. $\delta_{def}^{abc} = \delta_{ghi}^{abc} \delta_{def}^{ghi}$ and likewise for the projectors δ_{cd}^{ab} and δ_{efgh}^{abcd} :

$$\delta_{b_1 b_2 \dots b_n}^{a_1 a_2 \dots a_n} = \begin{cases} +1/n! & \text{for } b_1 b_2 \dots b_n \text{ an even permutation of } a_1 a_2 \dots a_n, \\ -1/n! & \text{for } b_1 b_2 \dots b_n \text{ an odd permutation of } a_1 a_2 \dots a_n, \\ 0 & \text{else} \end{cases} \tag{3}$$

3 Previously known stationary points

The potential $V(\phi_{\mathbf{n}})$ defined in (2) is known to have a local maximum at $\phi_{\mathbf{n}} = 0$ with $V = -6g^2$ of maximal symmetry $SO(8)$ and unbroken $\mathcal{N} = 8$ supersymmetry. In addition to this, six further stationary points in this potential have been published, which are also shown in table 2. Their detailed locations are given in [18, 19]. Numerical data are given in Appendix B in particular to simplify matching the conventions used in this work against conventions used in other parts of the literature.

Nr.	$V(\phi_n)/g^2$	approx.	Residual gauge group	Residual SUSY
0	- 6	- 6	$SO(8)$	$\mathcal{N} = 8$
1	$-2 \cdot 5^{3/4}$	- 6.687403	$SO(7)_s$	—
2	$-5^{5/2} \cdot 2^{-3}$	- 6.987712	$SO(7)_c$	—
3	$-2^{7/2} \cdot 3^{13/4} \cdot 5^{-5/2}$	- 7.191576	G_2	$\mathcal{N} = 1$
4	$-3^{5/2} \cdot 2^{-1}$	- 7.794229	$SU(3) \times U(1)$	$\mathcal{N} = 2$
5	- 8	- 8	$SU(4)$	—
6	-14	-14	$SO(3) \times SO(3)$	—

Table 2: The previously known stationary points

Nr.	approx. $V(\phi_n)/g^2$	Residual gauge group	Residual SUSY
7	- 9.987083	$U(1)$	—
8	-10.434713	—	—
9	-10.674754	$U(1) \times U(1)$	—
10	-11.656854	$U(1) \times U(1)$	—
11	-12.000000	$U(1) \times U(1)$	$\mathcal{N} = 1?$
12	-13.623653	$U(1)$	—
13	-13.676114	—	—
14	-14.970385	$U(1)$	—
15	-16.414456	—	—
16	-17.876428	—	—
17	-18.052693	—	—
18	-21.265976	—	—
19	-21.408498	—	—
20	-25.149369	—	—

Table 3: The new stationary points

4 New stationary points

Employing the sensitivity backpropagation method presented in [22], which already demonstrated its utility for the study of the somewhat more involved E_8 potential of $\mathcal{N} = 16$ three-dimensional Chern-Simons supergravity with gauge group $SO(8) \times SO(8)$, strong numerical evidence for a number of further stationary points in the potential could be obtained. Here, it is important to point out that, while more than doubling the previously known amount of data, the new stationary points listed below perhaps are just a comparatively small selection of the totality of further solutions. While the present article demonstrates the utility of this numerical technique, a more complete analysis will be deferred to a subsequent publication that also addresses a number of open issues with respect to more convenient and hence usable presentations of solutions, as well as semi-automatic derivation of robust analytical expressions (guided by numerical input). Essentially, it is feasible to use heuristics based on numerical observations to set up algebraic equation systems that both can be solved exactly and also stringently checked against the stationarity conditions. This is briefly explained in the last section of [28]. The new stationary points are given in table 3.

Detailed information on their location as well as their residual symmetries and fermion masses are given in Appendix B.

5 Details on the calculation

For these investigations, the “misalignment function” on the scalar manifold to be minimized numerically has been taken to be the length-squared of the potential’s gradient, rather than an explicit quadratic stationarity condition in the A -tensors, as in [22]. The corresponding A_1/A_2 stationarity condition is listed nevertheless (cf. e.g. (2.21) and (2.22), as well as the associated discussion in [24]) and has been used as an independent check to numerically validate the new solutions:

$$\begin{aligned} Q^{ijkl} &= -\frac{1}{24}\epsilon^{ijklmnpq}Q_{mnpq} \\ \text{with} \\ Q^{ijkl} &= \left(\frac{3}{4}A_{2m}{}^{ni'j'}A_{2n}{}^{k'l'm} - A_1{}^{mi'}A_{2m}{}^{j'k'l'}\right)\delta_{i'j'k'l'} \end{aligned} \quad (4)$$

This decision was motivated in part by the desire to learn whether the computation of quantities involving second derivatives would be feasible using the sensitivity backpropagation method, and how good its performance would be. As it turns out, the computations for this potential on average take considerably longer than the corresponding calculations for the E_8 potential of three-dimensional maximal gauged supergravity, but still are quite feasible even on a single notebook computer. Also, it is noteworthy that, with this approach, the time needed to obtain a solution as well as the numerical accuracy that is easily obtainable vary much stronger than in the three-dimensional case studied in [22]. This might in part be attributed to the potential being a sixth-order polynomial in the entries of the T -tensor here, while it is only fourth order in $D = 3$. While the superiority of this numerical method for identifying stationary points is again clearly demonstrated, a subsequent more detailed analysis that intends to determine a large number of solutions should perhaps use code that is better optimized and uses the Q -tensor criterion. For this reason, no code is yet included in the arxiv.org preprint upload of this work.

A technical issue arises concerning the presentation of results: typically, a solution found numerically will initially be given in a fairly awkward way, a more or less generic numerical vector of length 70. In many cases, in particular in situations with large residual gauge symmetries, this may be simplified considerably by choosing a suitable coordinate basis (i.e. finding an appropriate $SO(8)$ rotation) that sets many of the entries to zero. Due to the ‘diagonal traceless’ parts of the $\mathbf{35}_s$ and $\mathbf{35}_c$ representations, this is most easily implemented in the language of four-forms, obtained by³:

$$\phi_{abcd} = \sum_{n=1}^{35} S^{(SO(8))abcd}{}_{(\alpha\beta)} \phi_n \delta_{(\alpha\beta)}^n + \sum_{n=36}^{70} C^{(SO(8))abcd}{}_{(\dot{\alpha}\dot{\beta})} \phi_n \delta_{(\dot{\alpha}\dot{\beta})}^{n-35} \quad (5)$$

³This fully real parametrization differs from the usual (complex) four-form parametrization, but for the objective of finding a rotation that sets many entries to zero, this does not make much of a difference. This particular choice makes the sensitivity backpropagation code a bit easier to implement.

The “niceness” $N(\phi)$ of a presentation of a 70-vector is then obtained by taking:

$$N(\phi) = \sum_{abcd} (\phi_{abcd})^4. \quad (6)$$

Using the sensitivity backpropagation method again, a $SO(8)$ rotation that maximizes this niceness can be found quite efficiently. This ‘niceness’ function has no physical significance and is inspired by the simple observation that, when rotating a 2-dimensional vector (v_1, v_2) whose direction can be chosen freely, setting one coordinate to zero is equivalent to maximizing $v_1^4 + v_2^4$ (any even positive exponent larger than 2 would also do).

It is somewhat puzzling, however, to observe that, while this strategy clearly manages to simplify results, it also seems to often fail to come up with the best possible presentation here. In particular, when doing the numerical analysis, it seemed to produce one fairly awkward presentation for the $SU(4)$ -symmetric stationary point as it was re-discovered using these methods. Hence, a more exhaustive analysis should also pay more attention to this issue.

6 Discussion and Outlook

While having only numerical data on the locations of these new stationary points clearly is somewhat unsatisfying, it is equally clear that these numerical techniques provide highly valuable clues that should make it possible to analytically determine the exact properties of many of the new solutions. Taking as a specific example the numerical data on the location of the stationary point with $V/g^2 = -13.676114$, *which does not seem to have any residual gauge symmetry*, an ansatz such this one seems highly suggestive:

$$\begin{aligned} -\phi_{[1257]+} &= +\phi_{[1457]+} = -\phi_{[2368]+} = +\phi_{[3468]+} = A, \\ -\phi_{[1348]+} &= +\phi_{[1478]+} = +\phi_{[2356]+} = -\phi_{[2567]+} = B, \\ +\phi_{[1568]+} &= +\phi_{[2347]+} = C, \\ +\phi_{[1235]+} &= -\phi_{[1345]+} = +\phi_{[2678]+} = -\phi_{[4678]+} = D, \\ -\phi_{[1246]+} &= +\phi_{[1367]+} = -\phi_{[2458]+} = +\phi_{[3578]+} = E, \\ +\phi_{[1235]-} &= -\phi_{[1345]-} = -\phi_{[2678]-} = +\phi_{[4678]-} = F, \\ -\phi_{[1348]-} &= -\phi_{[1478]-} = +\phi_{[2356]-} = +\phi_{[2567]-} = G, \\ +\phi_{[1257]-} &= -\phi_{[1457]-} = -\phi_{[2368]-} = +\phi_{[3468]-} = H, \\ -\phi_{[1246]-} &= +\phi_{[1367]-} = +\phi_{[2458]-} = -\phi_{[3578]-} = I, \\ -\phi_{[1238]-} &= -\phi_{[1278]-} = +\phi_{[3456]-} = +\phi_{[4567]-} = J, \\ -\phi_{[1238]+} &= +\phi_{[1278]+} = +\phi_{[3456]+} = -\phi_{[4567]+} = K, \end{aligned} \quad (7)$$

$$\begin{aligned} A &\approx 0.0210, B \approx 0.0403, C \approx 0.0863, D \approx 0.0961, \\ E &\approx 0.1024, F \approx 0.1221, G \approx 0.1488, H \approx 0.1970, \\ I &\approx 0.3271, J \approx 0.6484, K \approx 0.6513. \end{aligned}$$

This alone considerably reduces the complexity of the problem. Here, one has to keep in mind that the large number of stationary points to be investigated strongly favours (semi-)automizable strategies over strategies that involve manual computations. A promising route to continue from here is to next determine the parameters $A - K$ to high accuracy, using either fast high-precision floatingpoint arithmetics as provided e.g. by [25], or more conventional implementations of multiprecision arithmetics. This then should suffice to derive additional

hypotheses on algebraic relations between these parameters. While this should further reduce the number of variables and hence simplify analytic matrix exponentiation, we note in passing that full analytic (‘Euler angle’) parametrizations of nine-dimensional submanifolds of E_7 have already been technically feasible seven years ago. Likewise, hybrid symbolic-numeric techniques can be employed to automatically generate and verify conjectures for algebraic relations between the entries of A_1 and A_2 , and hence ultimately the matrix $\mathcal{V}^{\tilde{A}}_{\tilde{B}}$.

Therefore, a complete determination of the exact properties of a substantial fraction of all the stationary points of the scalar potential of $\mathcal{N} = 8$, $D = 4$, $SO(8)$ -gauged supergravity should have come within technological reach now.

Considering the specific data that have been produced in this investigation, the most exciting outcome is of course the numerical evidence for a new vacuum with residual $U(1) \times U(1)$ gauge symmetry that seems to have a residual $\mathcal{N} = 1$ supersymmetry. Using numerical data for the A_1 and A_2 tensors that can be obtained with the code provided in [28], this can be seen in two ways: on the one hand, the A_1 tensor is almost diagonal, with just a 2×2 block in the (7,8)-coordinates that is of the form:

$$M = \begin{pmatrix} 2.1213251 & 0.7071019 \\ 0.7071019 & 2.1213251 \end{pmatrix} \quad (8)$$

Hence, the vector ϵ with $\epsilon_j = \delta_j^7 - \delta_j^8$ gets mapped to 1.4142232ϵ ; the square of this eigenvalue is 2.0000273 . According to [26] (see also (3.13)–(3.15) in [13]), any eigenvector of A_1 for which the corresponding eigenvalue λ satisfies $\lambda^2 = -V/6g^2$ (here: $\lambda^2 = 2$) corresponds to a residual supersymmetry. Also, the approximate numerical data given for A_2 seem to support $\epsilon_i A_2^i{}_{jkl} = 0$, hence providing independent numerical evidence.

The numerical data available so far already suggests a fairly simple structure that might be the basis of an analytic investigation:

$$\begin{aligned} -\phi_{[1357] +} &= -\phi_{[1468] +} = +\phi_{[2357] +} = -\phi_{[2468] +} = \\ &= -\phi_{[1357] -} = -\phi_{[1468] -} = -\phi_{[2357] -} = +\phi_{[2468] -} = A, \\ \\ +\phi_{[1367] +} &= -\phi_{[1458] +} = -\phi_{[2367] +} = -\phi_{[2458] +} = \\ &= -\phi_{[1367] -} = +\phi_{[1458] -} = -\phi_{[2367] -} = -\phi_{[2458] -} = B, \\ \\ -\phi_{[1278] -} &= +\phi_{[3456] -} = C \\ \\ A &\approx 0.0013, B \approx 0.5731, C \approx 0.6585 \end{aligned} \quad (9)$$

Further work on this solution will show whether this simple three-parameter ansatz can be established analytically.

Appendix A $E_{7(7)}$ conventions

We start the triality-inspired construction of explicit $E_{7(7)}$ generators by explicitly stating the entries of the $so(8)$ invariants $\gamma^a_{\alpha\dot{\alpha}}$. The choice given here reproduces the conventions in [27]: the nonzero entries of $\gamma^a_{\alpha\dot{\alpha}}$ are listed in compact notation as PQR_{\pm} , meaning $\gamma^{a=P}_{\alpha=Q \dot{\alpha}=R} = \pm 1$:

$$\begin{array}{cccccccc}
118_+ & 127_- & 136_- & 145_+ & 154_- & 163_+ & 172_+ & 181_- \\
212_+ & 221_- & 234_- & 243_+ & 256_+ & 265_- & 278_- & 287_+ \\
315_+ & 326_- & 337_+ & 348_- & 351_- & 362_+ & 373_- & 384_+ \\
413_+ & 424_+ & 431_- & 442_- & 457_- & 468_- & 475_+ & 486_+ \\
514_+ & 523_- & 532_+ & 541_- & 558_+ & 567_- & 576_+ & 585_- \\
616_+ & 625_+ & 638_+ & 647_+ & 652_- & 661_- & 674_- & 683_- \\
717_+ & 728_+ & 735_- & 746_- & 753_+ & 764_+ & 771_- & 782_- \\
811_+ & 822_+ & 833_+ & 844_+ & 855_+ & 866_+ & 877_+ & 888_+
\end{array} \tag{A.10}$$

From these, we then construct:

$$\gamma_{\alpha\beta}^{abcd} = \gamma_{\alpha\dot{\gamma}}^e \gamma_{\gamma\dot{\gamma}}^f \gamma_{\gamma\epsilon}^g \gamma_{\beta\epsilon}^h \delta_{efgh}^{abcd} \tag{A.11}$$

$$\gamma_{\dot{\alpha}\dot{\beta}}^{abcd} = \gamma_{\gamma\dot{\alpha}}^e \gamma_{\gamma\dot{\gamma}}^f \gamma_{\epsilon\dot{\gamma}}^g \gamma_{\epsilon\dot{\beta}}^h \delta_{efgh}^{abcd} \tag{A.12}$$

With these conventions, the tensor entries of $\gamma_{\alpha\beta}^{abcd}$ and $\gamma_{\dot{\alpha}\dot{\beta}}^{abcd}$ are ± 1 and 0.

The choice of basis for the 133-dimensional $E_{7(7)}$ Lie-algebra is such that the last 28 basis elements (i.e. elements 105–133) spawn the subalgebra of $SO(8)$. Elements 1–35 transform under this $SO(8)$ as symmetric traceless (s.t.) matrices over the spinors, elements 36–70 as s.t. matrices over the co-spinors, and elements 71–105 as s.t. matrices over the vectors. Hence, elements 71–133 form the subalgebra of $SU(8)$, and elements 1–70 are the ‘boost’ generators associated with the 70 scalars of supergravity that live on the coset manifold $E_{7(7)}/SU(8)$. Hence, an E_7 adjoint index \mathcal{A} splits as:

$$\mathcal{A} \rightarrow \underline{(\alpha\beta)} + \underline{(\dot{\alpha}\dot{\beta})} + \underline{(ij)} + \underline{[ij]} \tag{A.13}$$

We furthermore choose the basis for $SO(8)$ in such a way that element $105 + n$, when acting on the vector representation of $SO(8)$, would be represented as the rotation matrix $(R_{[jk]})^m_n = \delta_m^j \delta_n^k - \delta_n^j \delta_m^k$, i.e. the rotation that takes the k -direction into the j -direction, with $1 \leq j \leq k \leq 8$, $n = (j-1) \cdot 8 + (k-1) - (j-1)j/2$. Hence, E_7 basis element #106 corresponds to the rotation $R_{[12]}$, basis element 107 = $R_{[13]}$, etc. (lexicographically).

For the three 35-dimensional symmetric traceless vector/spinor/co-spinor representations, we use the convention that the first 7 basis elements correspond to the diagonal matrices $\text{diag}(1, -1, 0, \dots, 0)$, $\text{diag}(0, 1, -1, 0, \dots, 0)$, $\text{diag}(0, \dots, 0, 1, -1)$ (in that order), while element $7 + n$ corresponds to the matrix $(S_{(jk)})^m_p = \delta_m^j \delta_p^k + \delta_p^j \delta_m^k$, again with $1 \leq j \leq k \leq 8$, $n = (j-1) \cdot 8 + (k-1) - (j-1)j/2$ – and with a likewise lexicographical order for the corresponding non-diagonal parts of $S_{(\alpha\beta)}$ and $S_{(\dot{\alpha}\dot{\beta})}$. These conventions ensure that the E_7 symmetric bilinear form obtained from the fundamental representation, $G_{\mathcal{A}\mathcal{B}} = T_{\mathcal{A}\tilde{\mathcal{C}}}^{\tilde{\mathcal{D}}} T_{\mathcal{B}\tilde{\mathcal{D}}}^{\tilde{\mathcal{C}}}$ is almost diagonal, with entries +96 for $\tilde{\mathcal{B}} = \tilde{\mathcal{C}} \leq 70$, entries -96 for $\tilde{\mathcal{B}} = \tilde{\mathcal{C}} > 70$, entries -48 for the non-orthogonal generators corresponding to the diagonal parts of the symmetric traceless matrices over the spinors and co-spinors (i.e. $G_{\mathcal{A}=1\mathcal{B}=2}$, $G_{\mathcal{A}=4\mathcal{B}=3}$, $G_{\mathcal{A}=36\mathcal{B}=37}$, etc.), and entries +48 for $G_{\mathcal{A}=71\mathcal{B}=72}$, etc., which come from the diagonal part of the $\mathbf{35}_v$ representation.

In order to obtain explicit complex 56×56 matrices for E_7 generators in the fundamental representation, we first define the tensors below (where the

Einstein summation convention is suspended for ‘technical’ auxiliary indices that are set in typewriter font and do not belong to irreducible representations). Note that the adjoint representation of $SU(8)$ in the “bottom right” 28×28 matrix block involves complex conjugation, at variance with some of the earlier literature on this subject. Without this, the 133-dimensional algebra would not close. Evidently, a convenient choice for a Cartan subalgebra of this E_7 algebra is given by the generators #71, ..., #77. Given $T^{(E7)}_{\mathcal{A}\tilde{\mathcal{B}}}^{\tilde{\mathcal{C}}}$, the E_7 generator matrices g used to define the scalar potential are $(g^{(\mathbf{n})})^{\tilde{\mathcal{C}}}_{\tilde{\mathcal{B}}} = (T^{(E7)}_{\mathcal{A}=\mathbf{n}})^{\tilde{\mathcal{C}}}_{\tilde{\mathcal{B}}}$ for $n = 1, \dots, 70$.

With our conventions, the structure of the 56×56 generator matrices $(g^{(\mathbf{n})})$ is as follows: For $\mathbf{n} \leq 35$, each matrix has either zero or two non-zero entries per row which then are either $+1$ or -1 , the total number of non-zero rows being 48. For $35 < \mathbf{n} \leq 70$, each matrix likewise has 48 rows with two non-zero entries each that are either $+i$ or $-i$. For $70 < \mathbf{n} \leq 105$, each matrix has 24 non-zero entries in total, which each are either $+2i$ or $-2i$. Finally, for $\mathbf{n} > 105$, each matrix has 24 non-zero entries, with each of them being either $+2$ or -2 . In total, there are $70 \cdot 48 \cdot 2 + 63 \cdot 24 = 8232$ non-zero entries in the tensor $T^{(E7)}$.

$$\begin{aligned}
T^{(SU(8))}_{A_j^k} &= \begin{cases} +i & \text{for } A = j = k \leq 7 \\ -i & \text{for } A + 1 = j = k \leq 7 \\ +i & \text{for } 7 < A \leq 35, (\mathfrak{m}, \mathfrak{n}) = Z(A - 7), j = \mathfrak{m}, k = \mathfrak{n} \\ +i & \text{for } 7 < A \leq 35, (\mathfrak{m}, \mathfrak{n}) = Z(A - 7), j = \mathfrak{n}, k = \mathfrak{m} \\ +1 & \text{for } 35 < A, (\mathfrak{m}, \mathfrak{n}) = Z(A - 35), j = \mathfrak{m}, k = \mathfrak{n} \\ -1 & \text{for } 35 < A, (\mathfrak{m}, \mathfrak{n}) = Z(A - 35), j = \mathfrak{n}, k = \mathfrak{m} \end{cases} \\
S^{(SO(8))}_{(\underline{\alpha\beta})}{}^{abcd} &= \begin{cases} \gamma_{\alpha\beta}^{abcd}(\delta_{\mathfrak{m}}^{\alpha}\delta_{\mathfrak{m}}^{\beta} - \delta_{\mathfrak{n}}^{\alpha}\delta_{\mathfrak{n}}^{\beta}) & \text{for } \underline{(\alpha\beta)} = \mathfrak{m} = \mathfrak{n} - 1 \leq 7 \\ \gamma_{\alpha\beta}^{abcd}(\delta_{\mathfrak{m}}^{\alpha}\delta_{\mathfrak{n}}^{\beta} + \delta_{\mathfrak{n}}^{\alpha}\delta_{\mathfrak{m}}^{\beta}) & \text{for } \underline{(\alpha\beta)} > 7, (\mathfrak{m}, \mathfrak{n}) = Z(\underline{(\alpha\beta)} - 7) \end{cases} \\
C^{(SO(8))}_{(\underline{\dot{\alpha}\dot{\beta}})}{}^{abcd} &= \begin{cases} \gamma_{\dot{\alpha}\dot{\beta}}^{abcd}(\delta_{\mathfrak{m}}^{\dot{\alpha}}\delta_{\mathfrak{m}}^{\dot{\beta}} - \delta_{\mathfrak{n}}^{\dot{\alpha}}\delta_{\mathfrak{n}}^{\dot{\beta}}) & \text{for } \underline{(\dot{\alpha}\dot{\beta})} = \mathfrak{m} = \mathfrak{n} - 1 \leq 7 \\ \gamma_{\dot{\alpha}\dot{\beta}}^{abcd}(\delta_{\mathfrak{m}}^{\dot{\alpha}}\delta_{\mathfrak{n}}^{\dot{\beta}} + \delta_{\mathfrak{n}}^{\dot{\alpha}}\delta_{\mathfrak{m}}^{\dot{\beta}}) & \text{for } \underline{(\dot{\alpha}\dot{\beta})} > 7, (\mathfrak{m}, \mathfrak{n}) = Z(\underline{(\dot{\alpha}\dot{\beta})} - 7) \end{cases} \\
T^{(E7)}_{\mathcal{A}\tilde{\mathcal{B}}\tilde{\mathcal{C}}} &= \begin{cases} \frac{1}{8}S^{(SO(8))}_{(\underline{\alpha\beta})}{}^{abcd}(\delta_{\mathfrak{m}}^a\delta_{\mathfrak{n}}^b - \delta_{\mathfrak{n}}^a\delta_{\mathfrak{m}}^b)(\delta_{\mathfrak{p}}^c\delta_{\mathfrak{q}}^d - \delta_{\mathfrak{q}}^c\delta_{\mathfrak{p}}^d) & \text{for} \\ \mathcal{A} \leq 35, \underline{(\alpha\beta)} = \mathcal{A}, \\ \tilde{\mathcal{B}} > 28, (\mathfrak{p}, \mathfrak{q}) = Z(\tilde{\mathcal{B}} - 28), \\ \tilde{\mathcal{C}} \leq 28, (\mathfrak{m}, \mathfrak{n}) = Z(\tilde{\mathcal{C}}) \\ \hline \frac{1}{8}S^{(SO(8))}_{(\underline{\alpha\beta})}{}^{cdab}(\delta_{\mathfrak{m}}^c\delta_{\mathfrak{n}}^d - \delta_{\mathfrak{n}}^c\delta_{\mathfrak{m}}^d)(\delta_{\mathfrak{p}}^a\delta_{\mathfrak{q}}^b - \delta_{\mathfrak{q}}^a\delta_{\mathfrak{p}}^b) & \text{for} \\ \mathcal{A} \leq 35, \underline{(\alpha\beta)} = \mathcal{A}, \\ \tilde{\mathcal{B}} \leq 28, (\mathfrak{p}, \mathfrak{q}) = Z(\tilde{\mathcal{B}}), \\ \tilde{\mathcal{C}} > 28, (\mathfrak{m}, \mathfrak{n}) = Z(\tilde{\mathcal{C}} - 28) \\ \hline \frac{i}{8}C^{(SO(8))}_{(\underline{\dot{\alpha}\dot{\beta}})}{}^{abcd}(\delta_{\mathfrak{m}}^a\delta_{\mathfrak{n}}^b - \delta_{\mathfrak{n}}^a\delta_{\mathfrak{m}}^b)(\delta_{\mathfrak{p}}^c\delta_{\mathfrak{q}}^d - \delta_{\mathfrak{q}}^c\delta_{\mathfrak{p}}^d) & \text{for} \\ 35 < \mathcal{A} \leq 70, \underline{(\dot{\alpha}\dot{\beta})} = \mathcal{A} - 35, \\ \tilde{\mathcal{B}} > 28, (\mathfrak{p}, \mathfrak{q}) = Z(\tilde{\mathcal{B}} - 28), \\ \tilde{\mathcal{C}} \leq 28, (\mathfrak{m}, \mathfrak{n}) = Z(\tilde{\mathcal{C}}) \\ \hline -\frac{i}{8}C^{(SO(8))}_{(\underline{\dot{\alpha}\dot{\beta}})}{}^{cdab}(\delta_{\mathfrak{m}}^c\delta_{\mathfrak{n}}^d - \delta_{\mathfrak{n}}^c\delta_{\mathfrak{m}}^d)(\delta_{\mathfrak{p}}^a\delta_{\mathfrak{q}}^b - \delta_{\mathfrak{q}}^a\delta_{\mathfrak{p}}^b) & \text{for} \\ 35 < \mathcal{A} \leq 70, \underline{(\dot{\alpha}\dot{\beta})} = \mathcal{A} - 35, \\ \tilde{\mathcal{B}} \leq 28, (\mathfrak{p}, \mathfrak{q}) = Z(\tilde{\mathcal{B}}), \\ \tilde{\mathcal{C}} > 28, (\mathfrak{m}, \mathfrak{n}) = Z(\tilde{\mathcal{C}} - 28) \\ \hline 2T^{(SU(8))}_{A_j^k}(\delta_{\mathfrak{n}}^{\mathfrak{q}}\delta_{\mathfrak{m}}^j\delta_k^{\mathfrak{p}} + \delta_{\mathfrak{m}}^{\mathfrak{p}}\delta_{\mathfrak{n}}^j\delta_k^{\mathfrak{q}} - \delta_{\mathfrak{n}}^{\mathfrak{p}}\delta_{\mathfrak{m}}^j\delta_k^{\mathfrak{q}} - \delta_{\mathfrak{m}}^{\mathfrak{q}}\delta_{\mathfrak{n}}^j\delta_k^{\mathfrak{p}}) & \text{for} \\ \mathcal{A} > 70, A = \mathcal{A} - 70 \\ \tilde{\mathcal{B}} \leq 28, (\mathfrak{p}, \mathfrak{q}) = Z(\tilde{\mathcal{B}}), \\ \tilde{\mathcal{C}} \leq 28, (\mathfrak{m}, \mathfrak{n}) = Z(\tilde{\mathcal{C}}) \\ \hline 2(T^{(SU(8))}_{A_j^k})^*(\delta_{\mathfrak{n}}^{\mathfrak{q}}\delta_{\mathfrak{m}}^j\delta_k^{\mathfrak{p}} + \delta_{\mathfrak{m}}^{\mathfrak{p}}\delta_{\mathfrak{n}}^j\delta_k^{\mathfrak{q}} - \delta_{\mathfrak{n}}^{\mathfrak{p}}\delta_{\mathfrak{m}}^j\delta_k^{\mathfrak{q}} - \delta_{\mathfrak{m}}^{\mathfrak{q}}\delta_{\mathfrak{n}}^j\delta_k^{\mathfrak{p}}) & \text{for} \\ \mathcal{A} > 70, A = \mathcal{A} - 70 \\ \tilde{\mathcal{B}} > 28, (\mathfrak{p}, \mathfrak{q}) = Z(\tilde{\mathcal{B}} - 28), \\ \tilde{\mathcal{C}} > 28, (\mathfrak{m}, \mathfrak{n}) = Z(\tilde{\mathcal{C}} - 28) \end{cases} \tag{A.14}
\end{aligned}$$

Appendix B Numerical data

This section lists approximate numerical data both for the known as well as for the new solution. Each table gives:

- The identifier of the solution (matching tables 2 and 3)
- The value of the scalar potential V/g^2
- The ‘quality’ of the solution, $|Q|$ being the Frobenius norm of the Q -tensor and $|\nabla|$ being the length of the gradient (determined numerically via finite differences)⁴
- The residual symmetry and supersymmetry.
- The approximate location of the (degenerate) solution. For convenience, the data obtained using equations 5 and 6 have been post-processed into a more usual scalar(+)/pseudoscalar(-) coefficient notation.
- Approximate data on residual gauge group generators, for solutions with $U(1)^n$ symmetry.
- The (re-scaled) masses-squared of the fermions. For a (potentially complex) symmetric fermion mass matrix M , these are given by the eigenvalues of M^*M . All eigenvalues have been re-scaled by a factor $1/m_0^2 = -6/(V/g^2)$.

The gravitino masses-squared $(m^2/m_0^2)[\psi]$ are the eigenvalues of $A_1^*A_1$, while the spin-1/2 fermion masses-squared are the eigenvalues of $A_3^*A_3$, with A_3 being the matrix:

$$\begin{aligned} A_3[\underline{ijk}][\underline{lmn}] &= \frac{\sqrt{2}}{144}\epsilon_{ijkpqr}A_{2n}{}^{pqr}P_{[\underline{ijk}]}^{ijk}P_{[\underline{lmn}]}^{lmn} \\ &= -\frac{\sqrt{2}}{108}\epsilon_{ijkpqr}T_n{}^{pqr}P_{[\underline{ijk}]}^{ijk}P_{[\underline{lmn}]}^{lmn} \end{aligned} \quad (\text{B.15})$$

(Cf. eq. (2.14) in [24] and eq. (5.23) in [5]). Here, $P_{[\underline{ijk}]}^{ijk}$ is the $56 \times 8 \times 8 \times 8$ tensor that is fully anti-symmetric in ijk and has entries $\pm 1, 0$. Entries ± 1 occur for ijk being a cyclic/anti-cyclic permutation of the $[\underline{ijk}]$ -th index triplet without repetition in lexicographical order.

A companion article [28] provides numerical computer code containing more accurate data for the locations of all stationary points in the extended list, code to validate claims about stationarity, about residual (super-)symmetry, code to re-produce these tables, and also code to study the properties of these stationary points in more detail.

#0: $V/g^2 = -6.000000$, Quality: $ Q = 10^{-\infty}$, $ \nabla = 10^{-5.86}$ $\mathcal{N} = 8$
ϕ 0
Symmetry [28-dimensional]
$(m^2/m_0^2)[\psi]$ 1.000($\times 8$)
$(m^2/m_0^2)[\chi]$ 0.000($\times 56$)

⁴The validation code presented in [28] with which these tables were produced does not contain sensitivity backpropagation; accurate gradients that were determined via backpropagation show that the stationarity condition is satisfied even better than the numbers given here indicate. Floatingpoint accuracy sets a limit of $|\nabla| \geq 10^{-5.9}$ here, as can be seen from solution #0.

#1: $V/g^2 = -6.687403$, Quality: $ Q = 10^{-14.67}$, $ \nabla = 10^{-5.04}$	
ϕ	+0.2012 _{[1234] +} , -0.2012 _{[1256] +} , -0.2012 _{[1278] +} , -0.2012 _{[1358] +} , -0.2012 _{[1367] +} , +0.2012 _{[1457] +} , -0.2012 _{[1468] +} , +0.2012 _{[2357] +} , -0.2012 _{[2368] +} , +0.2012 _{[2458] +} , +0.2012 _{[2467] +} , -0.2012 _{[3456] +} , -0.2012 _{[3478] +} , +0.2012 _{[5678] +}
Symmetry [21-dimensional]	
$(m^2/m_0^2)[\psi]$	1.350 _($\times 8$)
$(m^2/m_0^2)[\chi]$	2.700 _($\times 8$) , 0.075 _($\times 48$)

#2: $V/g^2 = -6.987712$, Quality: $ Q = 10^{-14.30}$, $ \nabla = 10^{-5.06}$	
ϕ	+0.2406 _{[1234] -} , -0.2406 _{[1256] -} , +0.2406 _{[1278] -} , +0.2406 _{[1358] -} , -0.2406 _{[1367] -} , +0.2406 _{[1457] -} , +0.2406 _{[1468] -} , +0.2406 _{[2357] -} , +0.2406 _{[2368] -} , -0.2406 _{[2458] -} , +0.2406 _{[2467] -} , -0.2406 _{[3456] -} , +0.2406 _{[3478] -} , -0.2406 _{[5678] -}
Symmetry [21-dimensional]	
$(m^2/m_0^2)[\psi]$	1.350 _($\times 8$)
$(m^2/m_0^2)[\chi]$	2.700 _($\times 8$) , 0.075 _($\times 48$)

#3: $V/g^2 = -7.191576$, Quality: $ Q = 10^{-5.18}$, $ \nabla = 10^{-4.56}$, $\mathcal{N} = 1$	
ϕ	+0.1457 _{[1234] +} , -0.1457 _{[1256] +} , -0.1457 _{[1278] +} , -0.1457 _{[1358] +} , -0.1457 _{[1367] +} , +0.1457 _{[1457] +} , -0.1457 _{[1468] +} , +0.1457 _{[2357] +} , -0.1457 _{[2368] +} , +0.1457 _{[2458] +} , +0.1457 _{[2467] +} , -0.1457 _{[3456] +} , -0.1457 _{[3478] +} , +0.1457 _{[5678] +} , +0.2139 _{[1234] -} , -0.2139 _{[1256] -} , +0.2139 _{[1278] -} , +0.2139 _{[1358] -} , -0.2139 _{[1367] -} , +0.2139 _{[1457] -} , +0.2139 _{[1468] -} , +0.2139 _{[2357] -} , +0.2139 _{[2368] -} , -0.2139 _{[2458] -} , +0.2139 _{[2467] -} , -0.2139 _{[3456] -} , +0.2139 _{[3478] -} , -0.2139 _{[5678] -}
Symmetry [14-dimensional]	
$(m^2/m_0^2)[\psi]$	1.500 _($\times 7$) , 1.000
$(m^2/m_0^2)[\chi]$	3.000 _($\times 8$) , 0.750 _($\times 7$) , 0.083 _($\times 27$) , 0.000 _($\times 14$)

#4: $V/g^2 = -7.794229$, Quality: $ Q = 10^{-14.40}$, $ \nabla = 10^{-4.39}$, $\mathcal{N} = 2$	
ϕ	+0.2747 _{[1234] +} , +0.2747 _{[1256] +} , +0.2747 _{[1278] +} , +0.2747 _{[3456] +} , +0.2747 _{[3478] +} , +0.2747 _{[5678] +} , +0.3292 _{[1357] -} , -0.3292 _{[1368] -} , -0.3292 _{[1458] -} , -0.3292 _{[1467] -} , +0.3292 _{[2358] -} , +0.3292 _{[2367] -} , +0.3292 _{[2457] -} , -0.3292 _{[2468] -}
Symmetry [9-dimensional]	
$(m^2/m_0^2)[\psi]$	1.778 _($\times 6$) , 1.000 _($\times 2$)
$(m^2/m_0^2)[\chi]$	3.556 _($\times 6$) , 3.281 _($\times 2$) , 1.219 _($\times 2$) , 0.889 _($\times 12$) , 0.056 _($\times 18$) , 0.000 _($\times 16$)

#5: $V/g^2 = -8.000000$, Quality: $ Q = 10^{-14.52}$, $ \nabla = 10^{-4.49}$	
ϕ	+0.4407 _{[1357] -} , -0.4407 _{[1368] -} , -0.4407 _{[1458] -} , -0.4407 _{[1467] -} , +0.4407 _{[2358] -} , +0.4407 _{[2367] -} , +0.4407 _{[2457] -} , -0.4407 _{[2468] -}
Symmetry [15-dimensional]	

$(m^2/m_0^2)[\psi]$	1.688 _($\times 8$)
$(m^2/m_0^2)[\chi]$	3.375 _($\times 8$) , 2.344 _($\times 8$) , 0.094 _($\times 40$)

#6: $V/g^2 = -14.000000$, Quality: $ Q = 10^{-14.58}$, $ \nabla = 10^{-4.27}$	
ϕ	+1.0208 _{[1235] +} , -1.0208 _{[4678] +} , +1.0208 _{[1234] -} , -1.0208 _{[5678] -}
Symmetry	[6-dimensional]
$(m^2/m_0^2)[\psi]$	3.857 _($\times 2$) , 2.143 _($\times 6$)
$(m^2/m_0^2)[\chi]$	7.714 _($\times 2$) , 4.286 _($\times 12$) , 2.244 _($\times 18$) , 0.327 _($\times 18$) , 0.000 _($\times 6$)

#7: $V/g^2 = -9.987083$, Quality: $ Q = 10^{-5.77}$, $ \nabla = 10^{-4.26}$ $U(1)$	
ϕ	-0.1493 _{[1235] +} , -0.0094 _{[1238] +} , +0.1792 _{[1246] +} , +0.0237 _{[1257] +} , +0.5627 _{[1278] +} , -0.0648 _{[1345] +} , -0.1578 _{[1348] +} , +0.1719 _{[1367] +} , +0.2089 _{[1457] +} , -0.0651 _{[1478] +} , -0.1792 _{[1568] +} , -0.1792 _{[2347] +} , -0.0651 _{[2356] +} , -0.2089 _{[2368] +} , -0.1719 _{[2458] +} , -0.1578 _{[2567] +} , +0.0648 _{[2678] +} , +0.5627 _{[3456] +} , -0.0237 _{[3468] +} , -0.1792 _{[3578] +} , -0.0094 _{[4567] +} , +0.1493 _{[4678] +} , +0.4970 _{[1235] -} , -0.0599 _{[1238] -} , +0.1113 _{[1246] -} , -0.0772 _{[1257] -} , +0.0316 _{[1278] -} , -0.1270 _{[1345] -} , +0.4271 _{[1348] -} , -0.5403 _{[1367] -} , +0.0358 _{[1457] -} , +0.1338 _{[1478] -} , +0.1113 _{[1568] -} , -0.1113 _{[2347] -} , -0.1338 _{[2356] -} , +0.0358 _{[2368] -} , -0.5403 _{[2458] -} , -0.4271 _{[2567] -} , -0.1270 _{[2678] -} , -0.0316 _{[3456] -} , -0.0772 _{[3468] -} , +0.1113 _{[3578] -} , +0.0599 _{[4567] -} , +0.4970 _{[4678] -}
Symmetry	+0.399 R_{13} -0.088 R_{17} -0.361 R_{25} -0.435 R_{28} +0.088 R_{36} +0.376 R_{45} +0.454 R_{48} -0.399 R_{67}
$(m^2/m_0^2)[\psi]$	2.797 _($\times 2$) , 2.197 _($\times 2$) , 2.081, 1.890 _($\times 2$) , 1.598
$(m^2/m_0^2)[\chi]$	5.594 _($\times 2$) , 4.394 _($\times 2$) , 4.162, 4.116 _($\times 2$) , 3.861 _($\times 2$) , 3.781 _($\times 2$) , 3.770, 3.745, 3.196, 2.564 _($\times 2$) , 2.301, 1.896 _($\times 2$) , 1.454 _($\times 2$) , 1.400 _($\times 2$) , 1.381 _($\times 2$) , 1.375, 0.795 _($\times 2$) , 0.584, 0.508 _($\times 2$) , 0.225, 0.201 _($\times 2$) , 0.155, 0.153, 0.131 _($\times 2$) , 0.124 _($\times 2$) , 0.104 _($\times 2$) , 0.098 _($\times 2$) , 0.080 _($\times 2$) , 0.077, 0.049 _($\times 2$) , 0.040 _($\times 2$) , 0.038 _($\times 2$) , 0.029 _($\times 2$) , 0.017

#8: $V/g^2 = -10.434713$, Quality: $ Q = 10^{-7.38}$, $ \nabla = 10^{-4.41}$	
ϕ	+0.2101 _{[1235] +} , -0.2101 _{[1238] +} , +0.1924 _{[1246] +} , +0.4126 _{[1257] +} , +0.1426 _{[1278] +} , +0.0677 _{[1345] +} , +0.0677 _{[1348] +} , +0.2293 _{[1367] +} , +0.4126 _{[1457] +} , -0.1426 _{[1478] +} , -0.1924 _{[1568] +} , -0.1924 _{[2347] +} , -0.1426 _{[2356] +} , -0.4126 _{[2368] +} , -0.2293 _{[2458] +} , +0.0677 _{[2567] +} , -0.0677 _{[2678] +} , +0.1426 _{[3456] +} , -0.4126 _{[3468] +} , -0.1924 _{[3578] +} , -0.2101 _{[4567] +} , -0.2101 _{[4678] +} , +0.2807 _{[1235] -} , -0.2807 _{[1238] -} , -0.3141 _{[1246] -} , -0.4490 _{[1257] -} , +0.0339 _{[1278] -} , -0.1600 _{[1345] -} , -0.1600 _{[1348] -} , +0.4490 _{[1457] -} , +0.0339 _{[1478] -} , +0.3141 _{[1568] -} , -0.3141 _{[2347] -} , -0.0339 _{[2356] -} , +0.4490 _{[2368] -} , +0.1600 _{[2567] -} , -0.1600 _{[2678] -} , -0.0339 _{[3456] -} , -0.4490 _{[3468] -} , -0.3141 _{[3578] -} , +0.2807 _{[4567] -} , +0.2807 _{[4678] -}
$(m^2/m_0^2)[\psi]$	3.023, 2.620 _($\times 2$) , 2.330, 2.241, 1.951 _($\times 2$) , 1.651

$(m^2/m_0^2)[\chi]$	6.047, 5.240 _(×2) , 4.661, 4.483, 4.056, 4.005, 3.940 _(×2) , 3.901 _(×2) , 3.782, 3.628 _(×2) , 3.302, 3.035, 3.014, 2.576 _(×2) , 2.073, 1.644, 1.498 _(×2) , 1.395, 1.142, 1.052 _(×2) , 0.946, 0.896, 0.789 _(×2) , 0.648, 0.613, 0.463 _(×2) , 0.411, 0.140, 0.118 _(×2) , 0.107, 0.106, 0.081, 0.078, 0.071, 0.057 _(×2) , 0.049 _(×2) , 0.032 _(×2) , 0.031, 0.022, 0.006, 0.002 _(×2) , 0.001
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#9: $V/g^2 = -10.674754$, Quality: $ Q = 10^{-4.12}$, $ \nabla = 10^{-3.77}$ $U(1) \times U(1)$	
ϕ	+0.1682 _{[1235]+} , +0.2182 _{[1246]+} , -0.4136 _{[1248]+} , -0.0518 _{[1267]+} , -0.1797 _{[1278]+} , -0.0517 _{[1346]+} , -0.1801 _{[1348]+} , +0.2176 _{[1367]+} , -0.4137 _{[1378]+} , -0.1958 _{[1457]+} , -0.2316 _{[1568]+} , -0.2316 _{[2347]+} , +0.1958 _{[2368]+} , +0.4137 _{[2456]+} , -0.2176 _{[2458]+} , -0.1801 _{[2567]+} , -0.0517 _{[2578]+} , -0.1797 _{[3456]+} , -0.0518 _{[3458]+} , +0.4136 _{[3567]+} , -0.2182 _{[3578]+} , -0.1682 _{[4678]+} , +0.3291 _{[1246]-} , +0.3290 _{[1248]-} , -0.0003 _{[1267]-} , -0.0003 _{[1346]-} , -0.3287 _{[1367]-} , -0.3288 _{[1378]-} , +0.6578 _{[1457]-} , +0.6578 _{[2368]-} , -0.3288 _{[2456]-} , -0.3287 _{[2458]-} , +0.0003 _{[2578]-} , +0.0003 _{[3458]-} , +0.3290 _{[3567]-} , +0.3291 _{[3578]-}
Symmetry	-0.500 R_{12} +0.500 R_{25} -0.500 R_{67} +0.500 R_{78} ; -0.354 R_{14} -0.354 R_{17} +0.353 R_{26} +0.354 R_{28} -0.354 R_{36} +0.353 R_{38} -0.354 R_{45} -0.354 R_{57}
$(m^2/m_0^2)[\psi]$	2.656 _(×4) , 2.137 _(×4)
$(m^2/m_0^2)[\chi]$	5.312 _(×4) , 4.273 _(×4) , 3.995 _(×4) , 3.673 _(×4) , 3.463 _(×4) , 1.344 _(×4) , 1.195 _(×4) , 0.802 _(×4) , 0.638 _(×4) , 0.177 _(×4) , 0.090 _(×4) , 0.086 _(×4) , 0.085 _(×4) , 0.001 _(×4)

#10: $V/g^2 = -11.656854$, Quality: $ Q = 10^{-6.50}$, $ \nabla = 10^{-4.13}$ $U(1) \times U(1)$	
ϕ	-0.3330 _{[1238]+} , +0.3330 _{[1245]+} , +0.3330 _{[1356]+} , +0.3751 _{[1368]+} , +0.3330 _{[1468]+} , -0.3330 _{[2357]+} , +0.3751 _{[2457]+} , -0.3330 _{[2478]+} , +0.3330 _{[3678]+} , -0.3330 _{[4567]+} , -0.1841 _{[1235]-} , -0.3268 _{[1238]-} , -0.3268 _{[1245]-} , -0.5802 _{[1248]-} , +0.3268 _{[1356]-} , -0.5802 _{[1368]-} , -0.1841 _{[1456]-} , +0.3268 _{[1468]-} , +0.3268 _{[2357]-} , +0.1841 _{[2378]-} , +0.5802 _{[2457]-} , +0.3268 _{[2478]-} , -0.5802 _{[3567]-} , +0.3268 _{[3678]-} , +0.3268 _{[4567]-} , -0.1841 _{[4678]-}
Symmetry	-0.574 R_{18} -0.413 R_{24} +0.413 R_{36} -0.574 R_{57} ; -0.413 R_{18} +0.574 R_{24} -0.574 R_{36} -0.413 R_{57}
$(m^2/m_0^2)[\psi]$	2.561 _(×8)
$(m^2/m_0^2)[\chi]$	5.121 _(×8) , 3.685 _(×8) , 3.464 _(×8) , 0.854 _(×8) , 0.802 _(×8) , 0.118 _(×8) , 0.002 _(×8)

#11: $V/g^2 = -12.0$, Quality: $ Q = 10^{-6.09}$, $ \nabla = 10^{-4.29}$ $U(1) \times U(1)$ $\mathcal{N} = 1$	
ϕ	-0.0013 _{[1357]+} , +0.5731 _{[1367]+} , -0.5731 _{[1458]+} , -0.0013 _{[1468]+} , +0.0013 _{[2357]+} , -0.5731 _{[2367]+} , -0.5731 _{[2458]+} , -0.0013 _{[2468]+} , -0.6585 _{[1278]-} , -0.0013 _{[1357]-} , -0.5731 _{[1367]-} , +0.5731 _{[1458]-} , -0.0013 _{[1468]-} , -0.0013 _{[2357]-} , -0.5731 _{[2367]-} , -0.5731 _{[2458]-} , +0.0013 _{[2468]-} , +0.6585 _{[3456]-}
Symmetry	+0.707 R_{36} -0.707 R_{45} ; -0.707 R_{36} -0.707 R_{45}

$(m^2/m_0^2)[\psi]$	4.000, 3.000 _(×2) , 2.250 _(×4) , 1.000
$(m^2/m_0^2)[\chi]$	8.000, 6.000 _(×2) , 4.747 _(×4) , 4.500 _(×4) , 3.937 _(×4) , 3.732, 2.166 _(×4) , 2.000 _(×5) , 1.500 _(×2) , 1.125 _(×4) , 0.584 _(×4) , 0.500 _(×4) , 0.268, 0.125 _(×4) , 0.064 _(×4) , 0.003 _(×4) , 0.000 _(×4)

#12: $V/g^2 = -13.623653$, Quality: $ Q = 10^{-5.90}$, $ \nabla = 10^{-4.43}$ $U(1)$	
ϕ	+0.3511 _{[1234]+} , +0.5011 _{[1256]+} , -0.1538 _{[1278]+} , +0.0188 _{[1358]+} , +0.4933 _{[1367]+} , -0.4933 _{[1457]+} , +0.0188 _{[1468]+} , -0.0188 _{[2357]+} , +0.4933 _{[2368]+} , -0.4933 _{[2458]+} , -0.0188 _{[2467]+} , -0.1538 _{[3456]+} , +0.5011 _{[3478]+} , +0.3511 _{[5678]+} , +0.1309 _{[1234]-} , +0.5531 _{[1256]-} , +0.3862 _{[1278]-} , -0.1728 _{[1358]-} , +0.5789 _{[1367]-} , -0.5789 _{[1457]-} , -0.1728 _{[1468]-} , -0.1728 _{[2357]-} , -0.5789 _{[2368]-} , +0.5789 _{[2458]-} , -0.1728 _{[2467]-} , -0.3862 _{[3456]-} , -0.5531 _{[3478]-} , -0.1309 _{[5678]-}
Symmetry	+0.707 R_{34} +0.707 R_{56}
$(m^2/m_0^2)[\psi]$	3.425, 3.391, 3.377, 3.081 _(×2) , 2.307 _(×2) , 1.431
$(m^2/m_0^2)[\chi]$	6.850, 6.783, 6.755, 6.162 _(×2) , 5.621, 5.388, 4.865 _(×2) , 4.862 _(×2) , 4.614 _(×2) , 4.367, 4.271, 4.046, 3.403 _(×2) , 2.862, 2.756, 2.754 _(×2) , 2.284, 2.004 _(×2) , 1.937, 1.901 _(×2) , 1.553, 1.505 _(×2) , 1.350, 1.346 _(×2) , 1.272, 0.957 _(×2) , 0.540 _(×2) , 0.305, 0.242 _(×2) , 0.172 _(×2) , 0.088 _(×2) , 0.070, 0.067 _(×2) , 0.043 _(×2) , 0.028 _(×2) , 0.019, 0.008, 0.005

#13: $V/g^2 = -13.676114$, Quality: $ Q = 10^{-6.24}$, $ \nabla = 10^{-3.98}$	
ϕ	+0.0961 _{[1235]+} , -0.6513 _{[1238]+} , -0.1024 _{[1246]+} , -0.0210 _{[1257]+} , +0.6513 _{[1278]+} , -0.0961 _{[1345]+} , -0.0403 _{[1348]+} , +0.1024 _{[1367]+} , +0.0210 _{[1457]+} , +0.0403 _{[1478]+} , +0.0863 _{[1568]+} , +0.0863 _{[2347]+} , +0.0403 _{[2356]+} , -0.0210 _{[2368]+} , -0.1024 _{[2458]+} , -0.0403 _{[2567]+} , +0.0961 _{[2678]+} , +0.6513 _{[3456]+} , +0.0210 _{[3468]+} , +0.1024 _{[3578]+} , -0.6513 _{[4567]+} , -0.0961 _{[4678]+} , +0.1221 _{[1235]-} , -0.6484 _{[1238]-} , -0.3271 _{[1246]-} , +0.1970 _{[1257]-} , -0.6484 _{[1278]-} , -0.1221 _{[1345]-} , -0.1488 _{[1348]-} , +0.3271 _{[1367]-} , -0.1970 _{[1457]-} , -0.1488 _{[1478]-} , +0.1488 _{[2356]-} , -0.1970 _{[2368]-} , +0.3271 _{[2458]-} , +0.1488 _{[2567]-} , -0.1221 _{[2678]-} , +0.6484 _{[3456]-} , +0.1970 _{[3468]-} , -0.3271 _{[3578]-} , +0.6484 _{[4567]-} , +0.1221 _{[4678]-}
$(m^2/m_0^2)[\psi]$	3.620, 3.339 _(×2) , 3.220, 2.974, 2.612, 1.797 _(×2)
$(m^2/m_0^2)[\chi]$	7.239, 6.678 _(×2) , 6.440, 5.948, 5.241 _(×2) , 5.225, 5.116, 5.050, 4.783, 4.681, 4.456 _(×2) , 4.061 _(×2) , 3.594 _(×2) , 3.095 _(×2) , 2.898, 2.850, 2.780, 2.202, 2.189, 2.106 _(×2) , 1.852, 1.688, 1.551, 1.438 _(×2) , 1.204 _(×2) , 1.142, 1.129, 0.927, 0.582, 0.310 _(×2) , 0.285, 0.262, 0.223 _(×2) , 0.140, 0.089 _(×2) , 0.084, 0.081, 0.058, 0.035 _(×2) , 0.029, 0.006, 0.002 _(×2)

#14: $V/g^2 = -14.970385$, Quality: $ Q = 10^{-6.85}$, $ \nabla = 10^{-4.10}$ $U(1)$	
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ϕ	$+0.1728_{[1234]+}, +0.1197_{[1256]+}, +0.5219_{[1258]+}, -0.1197_{[1267]+},$ $-0.0480_{[1278]+}, -0.0480_{[1356]+}, +0.1197_{[1358]+}, -0.5219_{[1367]+},$ $+0.1197_{[1378]+}, +0.4398_{[1457]+}, -0.1728_{[1468]+}, +0.1728_{[2357]+},$ $-0.4398_{[2368]+}, -0.1197_{[2456]+}, +0.5219_{[2458]+}, -0.1197_{[2467]+},$ $+0.0480_{[2478]+}, -0.0480_{[3456]+}, -0.1197_{[3458]+}, +0.5219_{[3467]+},$ $+0.1197_{[3478]+}, +0.1728_{[5678]+}, +0.1364_{[1234]-}, +0.0967_{[1256]-},$ $-0.6919_{[1258]-}, -0.0967_{[1267]-}, +0.2809_{[1278]-}, +0.2809_{[1356]-},$ $+0.0967_{[1358]-}, +0.6919_{[1367]-}, +0.0967_{[1378]-}, -0.6119_{[1457]-},$ $-0.1364_{[1468]-}, -0.1364_{[2357]-}, -0.6119_{[2368]-}, +0.0967_{[2456]-},$ $+0.6919_{[2458]-}, +0.0967_{[2467]-}, +0.2809_{[2478]-}, -0.2809_{[3456]-},$ $+0.0967_{[3458]-}, +0.6919_{[3467]-}, -0.0967_{[3478]-}, -0.1364_{[5678]-}$
Symmetry	$-0.707 R_{28} \quad -0.707 R_{36}$
$(m^2/m_0^2)[\psi]$	$4.064_{(\times 2)}, 4.046, 3.965_{(\times 2)}, 3.626, 2.254, 1.568$
$(m^2/m_0^2)[\chi]$	$10.570, 10.564, 8.127_{(\times 2)}, 8.092, 7.930_{(\times 2)}, 7.251,$ $5.974_{(\times 2)}, 5.874_{(\times 2)}, 5.766, 5.731, 5.469, 5.396,$ $4.546_{(\times 2)}, 4.507, 4.458_{(\times 2)}, 3.661, 3.422, 3.136,$ $2.374_{(\times 2)}, 2.299_{(\times 2)}, 2.020_{(\times 2)}, 2.017, 1.602_{(\times 2)}, 1.594,$ $1.468_{(\times 2)}, 1.244_{(\times 2)}, 0.983, 0.578, 0.206_{(\times 2)}, 0.137_{(\times 2)},$ $0.082, 0.072_{(\times 2)}, 0.071_{(\times 2)}, 0.063_{(\times 2)}, 0.059_{(\times 2)}, 0.039,$ $0.024, 0.022$

#15: $V/g^2 = -16.414456$, Quality: $ Q = 10^{-5.04}, \nabla = 10^{-3.86}$	
ϕ	$-0.3317_{[1234]+}, +0.3194_{[1236]+}, +0.1187_{[1245]+}, +0.1116_{[1256]+},$ $-0.6446_{[1278]+}, -0.2277_{[1347]+}, -0.0186_{[1358]+}, -0.5445_{[1367]+},$ $+0.0629_{[1457]+}, -0.4439_{[1468]+}, -0.0817_{[1567]+}, +0.0817_{[2348]+},$ $+0.4439_{[2357]+}, -0.0629_{[2368]+}, +0.5445_{[2458]+}, +0.0186_{[2467]+},$ $+0.2277_{[2568]+}, -0.6446_{[3456]+}, +0.1116_{[3478]+}, +0.1187_{[3678]+},$ $+0.3194_{[4578]+}, -0.3317_{[5678]+}, +0.0699_{[1234]-}, +0.3235_{[1236]-},$ $-0.1401_{[1245]-}, +0.4283_{[1256]-}, +0.5643_{[1278]-}, -0.2090_{[1347]-},$ $-0.3835_{[1358]-}, +0.7370_{[1367]-}, -0.1472_{[1457]-}, +0.1624_{[1468]-},$ $-0.1029_{[1567]-}, -0.1029_{[2348]-}, +0.1624_{[2357]-}, -0.1472_{[2368]-},$ $+0.7370_{[2458]-}, -0.3835_{[2467]-}, -0.2090_{[2568]-}, -0.5643_{[3456]-},$ $-0.4283_{[3478]-}, +0.1401_{[3678]-}, -0.3235_{[4578]-}, -0.0699_{[5678]-}$
$(m^2/m_0^2)[\psi]$	$4.259, 4.083, 4.020, 3.729, 3.077, 3.042, 2.588, 2.289$
$(m^2/m_0^2)[\chi]$	$8.517, 8.166, 8.040, 7.458, 7.441, 7.384, 6.154, 6.085,$ $5.913, 5.756, 5.331, 5.252, 5.176, 5.086, 4.986, 4.860,$ $4.853, 4.579, 4.409, 4.325, 4.160, 4.077, 3.641, 3.262,$ $3.167, 3.134, 3.080, 2.881, 2.795, 2.702, 2.234, 2.183,$ $1.990, 1.801, 1.637, 1.242, 1.095, 0.872, 0.862, 0.800,$ $0.642, 0.605, 0.541, 0.505, 0.451, 0.389, 0.304, 0.229,$ $0.210, 0.206, 0.111, 0.082, 0.063, 0.025, 0.020, 0.017$

#16: $V/g^2 = -17.876443$, Quality: $ Q = 10^{-2.97}, \nabla = 10^{-2.26}$
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ϕ	$+0.4175_{[1234]+}, -0.0285_{[1235]+}, +0.0009_{[1236]+}, -0.0435_{[1237]+},$ $-0.0051_{[1238]+}, -0.0030_{[1245]+}, -0.0559_{[1246]+}, -0.0065_{[1247]+},$ $-0.0396_{[1248]+}, -0.0148_{[1256]+}, -0.0183_{[1257]+}, -0.0142_{[1258]+},$ $+0.0912_{[1267]+}, +0.0206_{[1268]+}, -0.7344_{[1278]+}, -0.0228_{[1345]+},$ $+0.0275_{[1346]+}, +0.0038_{[1347]+}, -0.0535_{[1348]+}, -0.0333_{[1356]+},$ $-0.1148_{[1357]+}, -0.0514_{[1358]+}, +0.6636_{[1367]+}, +0.1205_{[1368]+},$ $+0.0780_{[1378]+}, +0.0399_{[1456]+}, -0.0746_{[1457]+}, +0.1123_{[1458]+},$ $+0.1161_{[1467]+}, -0.3806_{[1468]+}, +0.0401_{[1478]+}, +0.0131_{[1567]+},$ $+0.0317_{[1568]+}, -0.0167_{[1578]+}, +0.0350_{[1678]+}, +0.0350_{[2345]+},$ $+0.0167_{[2346]+}, +0.0317_{[2347]+}, -0.0131_{[2348]+}, +0.0401_{[2356]+},$ $+0.3806_{[2357]+}, +0.1161_{[2358]+}, +0.1123_{[2367]+}, +0.0746_{[2368]+},$ $+0.0399_{[2378]+}, -0.0780_{[2456]+}, +0.1205_{[2457]+}, -0.6636_{[2458]+},$ $+0.0514_{[2467]+}, -0.1148_{[2468]+}, +0.0333_{[2478]+}, -0.0535_{[2567]+},$ $-0.0038_{[2568]+}, +0.0275_{[2578]+}, +0.0228_{[2678]+}, -0.7344_{[3456]+},$ $-0.0206_{[3457]+}, +0.0912_{[3458]+}, -0.0142_{[3467]+}, +0.0183_{[3468]+},$ $-0.0148_{[3478]+}, +0.0396_{[3567]+}, -0.0065_{[3568]+}, +0.0559_{[3578]+},$ $-0.0030_{[3678]+}, -0.0051_{[4567]+}, +0.0435_{[4568]+}, +0.0009_{[4578]+},$ $+0.0285_{[4678]+}, +0.4175_{[5678]+}, -0.0160_{[1234]-}, -0.0426_{[1235]-},$ $+0.0007_{[1236]-}, -0.0467_{[1237]-}, -0.0065_{[1238]-}, +0.0051_{[1245]-},$ $-0.0256_{[1246]-}, -0.0063_{[1247]-}, -0.1398_{[1248]-}, -0.0677_{[1256]-},$ $-0.0189_{[1257]-}, -0.3597_{[1258]-}, +0.0863_{[1267]-}, +0.0196_{[1268]-},$ $+0.7213_{[1278]-}, +0.0209_{[1345]-}, +0.1302_{[1346]-}, +0.0050_{[1347]-},$ $-0.0257_{[1348]-}, +0.3617_{[1356]-}, -0.1197_{[1357]-}, -0.0663_{[1358]-},$ $-0.7515_{[1367]-}, +0.1190_{[1368]-}, +0.0814_{[1378]-}, -0.0401_{[1456]-},$ $+0.0008_{[1457]-}, -0.1083_{[1458]-}, +0.1120_{[1467]-}, +0.0164_{[1468]-},$ $+0.0410_{[1478]-}, +0.0113_{[1567]-}, +0.0431_{[1568]-}, -0.0168_{[1578]-},$ $+0.0363_{[1678]-}, -0.0363_{[2345]-}, -0.0168_{[2346]-}, -0.0431_{[2347]-},$ $+0.0113_{[2348]-}, -0.0410_{[2356]-}, +0.0164_{[2357]-}, -0.1120_{[2358]-},$ $+0.1083_{[2367]-}, +0.0008_{[2368]-}, +0.0401_{[2378]-}, +0.0814_{[2456]-},$ $-0.1190_{[2457]-}, -0.7515_{[2458]-}, -0.0663_{[2467]-}, +0.1197_{[2468]-},$ $+0.3617_{[2478]-}, +0.0257_{[2567]-}, +0.0050_{[2568]-}, -0.1302_{[2578]-},$ $+0.0209_{[2678]-}, -0.7213_{[3456]-}, +0.0196_{[3457]-}, -0.0863_{[3458]-},$ $+0.3597_{[3467]-}, -0.0189_{[3468]-}, +0.0677_{[3478]-}, -0.1398_{[3567]-},$ $+0.0063_{[3568]-}, -0.0256_{[3578]-}, -0.0051_{[3678]-}, +0.0065_{[4567]-},$ $-0.0467_{[4568]-}, -0.0007_{[4578]-}, -0.0426_{[4678]-}, +0.0160_{[5678]-}$
	$(m^2/m_0^2)[\psi]$ 4.293 _($\times 2$) , 4.274 _($\times 2$) , 3.027 _($\times 2$) , 3.020 _($\times 2$)
	8.585 _($\times 2$) , 8.547 _($\times 2$) , 7.487 _($\times 2$) , 7.486 _($\times 2$) , 6.055 _($\times 2$) , 6.041 _($\times 2$) , 5.917 _($\times 2$) , 5.915 _($\times 2$) , 4.952 _($\times 2$) , 4.951 _($\times 2$) , $(m^2/m_0^2)[\chi]$ 4.377 _($\times 2$) , 4.372 _($\times 2$) , 3.665 _($\times 2$) , 3.652 _($\times 2$) , 2.109 _($\times 2$) , 2.102 _($\times 2$) , 1.559 _($\times 2$) , 1.558 _($\times 2$) , 1.356 _($\times 2$) , 1.351 _($\times 2$) , 0.818 _($\times 2$) , 0.813 _($\times 2$) , 0.700 _($\times 4$) , 0.199 _($\times 4$) , 0.029 _($\times 2$) , 0.028 _($\times 2$)

#17: $V/g^2 = -18.052693$, Quality: $|Q| = 10^{-5.88}$, $|\nabla| = 10^{-4.04}$

ϕ	$+0.1770_{[1235]+}, -0.1770_{[1236]+}, -0.1770_{[1245]+}, -0.1770_{[1246]+},$ $+0.3134_{[1278]+}, +0.2552_{[1348]+}, +0.6510_{[1357]+}, +0.6510_{[1367]+},$ $+0.0099_{[1457]+}, -0.0099_{[1467]+}, +0.2552_{[1568]+}, +0.2552_{[2347]+},$ $-0.0099_{[2358]+}, -0.0099_{[2368]+}, -0.6510_{[2458]+}, +0.6510_{[2468]+},$ $+0.2552_{[2567]+}, +0.3134_{[3456]+}, +0.1770_{[3578]+}, -0.1770_{[3678]+},$ $-0.1770_{[4578]+}, -0.1770_{[4678]+}, +0.0130_{[1235]-}, +0.0130_{[1236]-},$ $-0.0130_{[1245]-}, +0.0130_{[1246]-}, -0.1250_{[1348]-}, -0.8500_{[1357]-},$ $+0.8500_{[1367]-}, -0.3413_{[1457]-}, -0.3413_{[1467]-}, -0.1250_{[1568]-},$ $+0.1250_{[2347]-}, +0.3413_{[2358]-}, -0.3413_{[2368]-}, +0.8500_{[2458]-},$ $+0.8500_{[2468]-}, +0.1250_{[2567]-}, +0.0130_{[3578]-}, +0.0130_{[3678]-},$ $-0.0130_{[4578]-}, +0.0130_{[4678]-}$
$(m^2/m_0^2)[\psi]$	$4.398_{(\times 2)}, 3.811_{(\times 2)}, 3.346_{(\times 2)}, 2.359_{(\times 2)}$
$(m^2/m_0^2)[\chi]$	$8.796_{(\times 2)}, 7.623_{(\times 2)}, 6.923_{(\times 2)}, 6.900_{(\times 2)}, 6.692_{(\times 2)},$ $5.596_{(\times 2)}, 4.874_{(\times 2)}, 4.719_{(\times 2)}, 4.279_{(\times 2)}, 4.258_{(\times 2)},$ $4.176_{(\times 2)}, 4.036_{(\times 2)}, 3.482_{(\times 2)}, 3.200_{(\times 2)}, 2.963_{(\times 2)},$ $2.305_{(\times 2)}, 2.010_{(\times 2)}, 1.522_{(\times 2)}, 1.099_{(\times 2)}, 1.067_{(\times 2)},$ $1.060_{(\times 2)}, 0.924_{(\times 2)}, 0.440_{(\times 2)}, 0.126_{(\times 2)}, 0.070_{(\times 2)},$ $0.043_{(\times 2)}, 0.036_{(\times 2)}, 0.016_{(\times 2)}$

#18: $V/g^2 = -21.265976$, Quality: $ Q = 10^{-6.16}$, $ \nabla = 10^{-3.67}$	
ϕ	$-0.5622_{[1234]+}, +0.2761_{[1248]+}, -0.0763_{[1256]+}, +0.3668_{[1257]+},$ $-0.0763_{[1268]+}, -0.3668_{[1278]+}, +0.0447_{[1358]+}, +0.7687_{[1367]+},$ $-0.3668_{[1456]+}, -0.0763_{[1457]+}, +0.3668_{[1468]+}, -0.0763_{[1478]+},$ $+0.2761_{[1678]+}, +0.2761_{[2345]+}, -0.0763_{[2356]+}, -0.3668_{[2357]+},$ $+0.0763_{[2368]+}, -0.3668_{[2378]+}, -0.7687_{[2458]+}, -0.0447_{[2467]+},$ $-0.3668_{[3456]+}, +0.0763_{[3457]+}, -0.3668_{[3468]+}, -0.0763_{[3478]+},$ $-0.2761_{[3567]+}, -0.5622_{[5678]+}, -0.4045_{[1234]-}, -0.2766_{[1248]-},$ $+0.3801_{[1257]-}, +0.3801_{[1278]-}, -0.9522_{[1367]-}, +0.3801_{[1456]-},$ $+0.3801_{[1468]-}, +0.2766_{[1678]-}, -0.2766_{[2345]-}, +0.3801_{[2357]-},$ $-0.3801_{[2378]-}, -0.9522_{[2458]-}, -0.3801_{[3456]-}, +0.3801_{[3468]-},$ $-0.2766_{[3567]-}, +0.4045_{[5678]-}$
$(m^2/m_0^2)[\psi]$	$5.785_{(\times 2)}, 4.194_{(\times 4)}, 3.697_{(\times 2)}$
$(m^2/m_0^2)[\chi]$	$11.570_{(\times 2)}, 9.001_{(\times 2)}, 8.471_{(\times 4)}, 8.388_{(\times 4)}, 7.554_{(\times 2)},$ $7.394_{(\times 2)}, 7.065_{(\times 2)}, 6.352_{(\times 2)}, 6.220_{(\times 4)}, 4.597_{(\times 2)},$ $4.407_{(\times 4)}, 2.706_{(\times 2)}, 2.444_{(\times 2)}, 1.853_{(\times 4)}, 1.476_{(\times 2)},$ $1.331_{(\times 2)}, 1.303_{(\times 4)}, 0.693_{(\times 2)}, 0.569_{(\times 2)}, 0.293_{(\times 4)},$ $0.210_{(\times 2)}$

#19: $V/g^2 = -21.408498$, Quality: $ Q = 10^{-3.63}$, $ \nabla = 10^{-2.84}$	
ϕ	$+0.2178_{[1234]+}, -0.6540_{[1256]+}, -0.0068_{[1267]+}, +0.6212_{[1278]+},$ $-0.0068_{[1356]+}, -0.1884_{[1358]+}, -0.6540_{[1367]+}, +0.2178_{[1457]+},$ $-0.2704_{[1468]+}, +0.2704_{[2357]+}, -0.2178_{[2368]+}, +0.6540_{[2458]+},$ $+0.1884_{[2467]+}, +0.0068_{[2478]+}, +0.6212_{[3456]+}, -0.0068_{[3458]+},$ $-0.6540_{[3478]+}, +0.2178_{[5678]+}, +0.3332_{[1234]-}, +0.9511_{[1256]-},$ $+0.0004_{[1258]-}, -0.0055_{[1267]-}, -0.0055_{[1356]-}, +0.9511_{[1367]-},$ $-0.0004_{[1378]-}, -0.3332_{[1457]-}, -0.3332_{[2368]-}, -0.0004_{[2456]-},$ $+0.9511_{[2458]-}, -0.0055_{[2478]-}, +0.0055_{[3458]-}, -0.0004_{[3467]-},$ $-0.9511_{[3478]-}, -0.3332_{[5678]-}$

$(m^2/m_0^2)[\psi]$	5.055 _(×2) , 5.052 _(×2) , 3.957 _(×4)
$(m^2/m_0^2)[\chi]$	11.250 _(×4) , 10.109 _(×2) , 10.105 _(×2) , 7.914 _(×4) , 7.857 _(×4) , 5.251 _(×4) , 4.998 _(×4) , 4.973 _(×2) , 4.965 _(×2) , 4.459 _(×4) , 2.159 _(×2) , 2.153 _(×2) , 1.577 _(×2) , 1.574 _(×2) , 1.273 _(×4) , 0.883 _(×4) , 0.222 _(×4) , 0.181 _(×2) , 0.179 _(×2)

#20: $V/g^2 = -25.149369$, Quality: $ Q = 10^{-6.30}$, $ \nabla = 10^{-3.50}$	
ϕ	-0.1859 _{[1234] +} , +0.3286 _{[1256] +} , +0.0475 _{[1257] +} , -0.0475 _{[1268] +} , -0.3287 _{[1278] +} , -0.0852 _{[1356] +} , -0.3827 _{[1357] +} , +0.0062 _{[1358] +} , -0.2180 _{[1367] +} , -0.3826 _{[1368] +} , -0.0851 _{[1378] +} , -0.1129 _{[1456] +} , +0.1179 _{[1457] +} , -1.0129 _{[1458] +} , -0.2124 _{[1467] +} , +0.1181 _{[1468] +} , -0.1128 _{[1478] +} , -0.1128 _{[2356] +} , -0.1181 _{[2357] +} , -0.2124 _{[2358] +} , -1.0129 _{[2367] +} , -0.1179 _{[2368] +} , -0.1129 _{[2378] +} , +0.0851 _{[2456] +} , -0.3826 _{[2457] +} , +0.2180 _{[2458] +} , -0.0062 _{[2467] +} , -0.3827 _{[2468] +} , +0.0852 _{[2478] +} , -0.3287 _{[3456] +} , +0.0475 _{[3457] +} , -0.0475 _{[3468] +} , +0.3286 _{[3478] +} , -0.1859 _{[5678] +} , +0.1870 _{[1234] -} , -0.0641 _{[1256] -} , -0.4430 _{[1257] -} , +0.4430 _{[1268] -} , +0.0642 _{[1278] -} , -0.0403 _{[1356] -} , +0.7939 _{[1357] -} , +0.2410 _{[1358] -} , +0.2494 _{[1367] -} , +0.7937 _{[1368] -} , -0.0402 _{[1378] -} , -0.4082 _{[1456] -} , +0.0520 _{[1457] -} , -0.2165 _{[1458] -} , -0.0740 _{[1467] -} , +0.0520 _{[1468] -} , -0.4082 _{[1478] -} , +0.4082 _{[2356] -} , +0.0520 _{[2357] -} , +0.0740 _{[2358] -} , +0.2165 _{[2367] -} , +0.0520 _{[2368] -} , +0.4082 _{[2378] -} , -0.0402 _{[2456] -} , -0.7937 _{[2457] -} , +0.2494 _{[2458] -} , +0.2410 _{[2467] -} , -0.7939 _{[2468] -} , -0.0403 _{[2478] -} , -0.0642 _{[3456] -} , +0.4430 _{[3457] -} , -0.4430 _{[3468] -} , +0.0641 _{[3478] -} , -0.1870 _{[5678] -}
$(m^2/m_0^2)[\psi]$	7.557, 6.906, 5.728, 5.212, 4.283, 3.325, 3.265, 2.715
$(m^2/m_0^2)[\chi]$	15.114, 13.813, 13.636, 13.631, 11.455, 10.424, 9.730, 9.728, 9.169, 9.163, 8.566, 7.200, 7.106, 7.010, 6.888, 6.866, 6.761, 6.650, 6.530, 6.016, 6.015, 5.609, 5.432, 5.430, 5.379, 5.311, 5.268, 5.058, 4.398, 4.379, 4.106, 3.962, 3.869, 3.761, 3.544, 3.367, 2.365, 2.226, 2.181, 2.163, 1.437, 1.333, 1.232, 1.221, 0.952, 0.844, 0.816, 0.580, 0.376, 0.315, 0.237, 0.219, 0.053, 0.021, 0.002, 0.001

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